Lorenz-gauge reconstruction in electromagnetism with sources

Stephen R. Green
Albert Einstein Institute Postdam

with V. Toomani, S. Hollands, …

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Status of field reconstruction in Kerr

- Starting from Teukolsky solutions, use a Hertz potential to generate corresponding A_a or h_{ab}.

Maxwell

	Radiation gauge	Lorenz gauge
no sources	✓ CCK (1970s)	✓ Dolan (2019), Wardell, Kavanagh (2020)
sources	✓ Toomani, Hollands (2021)	This talk

Einstein

	Radiation gauge	Lorenz gauge
no sources	✓ CCK (1970s)	✗
sources	✓ Green, Hollands, Zimmerman (2020)	✗

Preliminaries: exterior calculus

- Differential p-form $\omega_{a_1 \ldots a_p} = \omega_{[a_1 \ldots a_p]}$
  - totally antisymmetric
- Wedge product $(\alpha \wedge \beta)_{ab} = 2\alpha_{[a}\beta_{b]}$
  - antisymmetrized product
- Exterior derivative $(d\omega)_{ba_1 \ldots a_p} = (p+1)\nabla_{[b}\omega_{a_1 \ldots a_p]}$
  - antisymmerized derivative; $d^2 = 0$
- Hodge dual $(*\alpha)_{abc} = \epsilon_{abc}{}^{d}\alpha_d$
  - $** = \pm 1$
- Co-differential $\delta = *d*$
  - divergence; $\delta^2 = 0$

- **Electromagnetism**
  - 2-form field F_{ab}
  - Maxwell equations

$$dF = 0$$

$$\delta F = j$$

  - Maxwell potential 1-form A_a

$$F = dA$$

Automatically $dF = d^2A = 0$

Bivector basis

- Decompose F_{ab}

Maxwell scalars

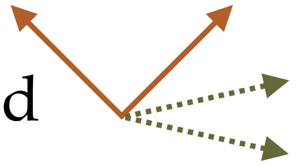
$$F = \phi_0 X + \phi_1 Y + \phi_2 Z + \text{c.c.}$$

$\bar{m} \wedge n$

$n \wedge l - \bar{m} \wedge m$

$l \wedge m$

$(l, n, \bar{m}, m)$ is NP tetrad



- X, Y, Z are "anti-self-dual": $\quad * X = iX$

- $\bar{X}, \bar{Y}, \bar{Z}$ are "self-dual": $\quad * \bar{X} = -i\bar{X}$

Exterior calculus and Teukolsky

- At level of forms, Teukolsky decoupling operator is simply $\mathcal{S} \equiv \zeta^{-2} d\zeta^{2}$,

$\zeta = \Psi_2^{-1/3}$

Teukolsky equations

$$\zeta^{-2} d\zeta^{2} \delta F^{A} = X\left[\frac{1}{2}\mathcal{O}(\phi_0)\right]$$

$\mathcal{O}$

Fackerell-Ipser equation

$$
\begin{aligned}
&+ Y\left[\frac{1}{2}\Theta^{a}\Theta_{a}\phi_1 + (B+B')\cdot\Theta\phi_1 + 2(B\cdot B')\phi_1 + 2\Psi_2\phi_1\right] \\
&+ Z\left[\frac{1}{2}\mathcal{O}'(\phi_2)\right] \\
&- \bar{X}\zeta^{-2}\left\{\eth'^{2}(\zeta^{2}\phi_0) + \text{þ}^{2}(\zeta^{2}\phi_2) - 2[\text{þ}\,\eth' - \tau'\,\text{þ} + (\text{þ}\,\tau') - (\eth'\,\rho)](\zeta^{2}\phi_1)\right\} \\
&- \bar{Y}\zeta^{-2}\left\{(\text{þ}'\,\eth' + \bar{\tau}\,\text{þ}')(\zeta^{2}\phi_0) + (\text{þ}\,\eth + \bar{\tau}'\,\text{þ})(\zeta^{2}\phi_2) - [\eth\,\eth' + \text{þ}\,\text{þ}' + \ldots](\zeta^{2}\phi_1)\right\} \\
&- \bar{Z}\zeta^{-2}\left\{\text{þ}'^{2}(\zeta^{2}\phi_0) + \eth^{2}(\zeta^{2}\phi_2) - 2[\text{þ}'\,\eth + \ldots](\zeta^{2}\phi_1)\right\}
\end{aligned}
$$

(related to)
Teukolsky-Starobinski identities

Exterior calculus and Teukolsky

- With $F = dA \equiv \mathscr{T} A$, the sourced Maxwell equation is

$$j = \delta F = \delta d A \equiv \mathscr{E} A$$

- In terms of operators on forms, the (electromagnetic) Wald identity is trivial:

$$\begin{array}{ccccccc} \zeta^{-2} d \zeta^{2} & \circ & \delta d & = & \zeta^{-2} d \zeta^{2} \delta & \circ & d \\ \mathscr{S} & & \mathscr{E} & & \mathscr{O} & & \mathscr{T} \end{array}$$

- This (and its adjoint) can now be used in reconstruction.

Lorenz-gauge reconstruction

- Source-free case (Dolan, Wardell, Kavanagh) easy using exterior calculus:
  - Hertz 2-form $\mathcal{H} = \Phi_0 X + \Phi_2 Z$

$$\longrightarrow \qquad A = \delta\zeta\mathcal{H}$$

  - Automatically Lorenz gauge $\delta A = \delta^2\zeta\mathcal{H} = 0$
  - Field strength

$$\begin{aligned} F &= dA \\ &= d\delta\zeta\mathcal{H} \\ &= \underbrace{\zeta^{-1}d\zeta^2}_{\zeta\mathcal{O}}\delta\mathcal{H} + \underbrace{X\text{Ł}_\xi\Phi_0 - Z\overset{\text{GHP-Lie derivative}}{\text{Ł}}_\xi\Phi_2 + Y(C'\cdot\Theta\Phi_0 + C\cdot\Theta\Phi_2)}_{\text{anti-self-dual}} \end{aligned}$$

Lorenz-gauge reconstruction

$$
\begin{aligned}
F &= dA \\
&= d\delta\zeta\mathcal{H} \\
&= \underbrace{\zeta^{-1}d\zeta^{2}\delta}_{\zeta\mathscr{O}}\mathcal{H} + \underbrace{X\text{Ł}_{\xi}\Phi_0 - Z\text{Ł}_{\xi}\Phi_2 + Y(C'\cdot\Theta\Phi_0 + C\cdot\Theta\Phi_2)}_{\text{anti-self-dual}}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{O}\mathcal{H} = {} & \frac{1}{2}X\mathcal{O}(\Phi_0) + \frac{1}{2}Z\mathcal{O}'(\Phi_2) \\
& - \bar{X}\zeta^{-2}\left[\eth'^{2}(\zeta^{2}\Phi_0) + \text{þ}^{2}(\zeta^{2}\Phi_2)\right] - \bar{Z}\zeta^{-2}\left[\text{þ}'^{2}(\zeta^{2}\Phi_0) + \eth^{2}(\zeta^{2}\Phi_2)\right] \\
& - \bar{Y}\zeta^{-2}\left[(\text{þ}'\,\eth' + \bar{\tau}\,\text{þ}')(\zeta^{2}\Phi_0) + (\text{þ}\,\eth + \bar{\tau}'\,\text{þ})(\zeta^{2}\Phi_2)\right]
\end{aligned}
$$

- If $(\Phi_0, -\Phi_2)$ satisfy TSI, then self-dual part of $\mathscr{O}\mathscr{H}$ vanishes, and F is anti-self-dual. Thus, $F = -i*F = -i*dA$, and

$$\delta F = *d*F = id^{2}A = 0 \quad \longrightarrow \quad \text{solution}$$

- Consistent with inversion relations $\text{Ł}_{\xi}\Phi_0 = \phi_0$

$$\text{Ł}_{\xi}\Phi_2 = -\phi_2$$

Inclusion of sources

- Suppose we have a Maxwell solution (F, j).
- Hertz potential stays the same

$$\mathcal{H} = \Phi_0 X + \Phi_2 Z$$

- Potential picks up a corrector 1-form

$$A = \delta\zeta\mathcal{H} + G$$

- Inversion relations

$$Ł_\xi \Phi_0 = \phi_0$$
$$Ł_\xi \Phi_2 = -\phi_2$$

$$Ł_\xi G_n = \zeta j_n$$
$$Ł_\xi G_l = -\zeta j_l$$
$$Ł_\xi G_m = -\zeta j_m$$
$$Ł_\xi G_{\bar{m}} = \zeta j_{\bar{m}}$$

- Similar (but more involved) calculation to source-free case.

Conclusions

- Nonzero charge current j can be included as a source for Lorenz gauge reconstruction in electromagnetism.
    - Corrector is related to j via time derivatives.
- Differential forms notation very useful in calculations.
    - **Next steps:** Other Lorenz gauge constructions. (Barry's talk)
    - **Next steps:** Gravitational case. Can the formalism be adapted? Take bivector components of Weyl tensor?
    - **Next steps:** Point particle. Reconstruct field.
- If successful, this should greatly simplify self-force calculations, by enabling use of Teukolsky equation in producing Lorenz-gauge metric perturbations.

Thank You!