Teukolsky formalism for nonlinear Kerr perturbations

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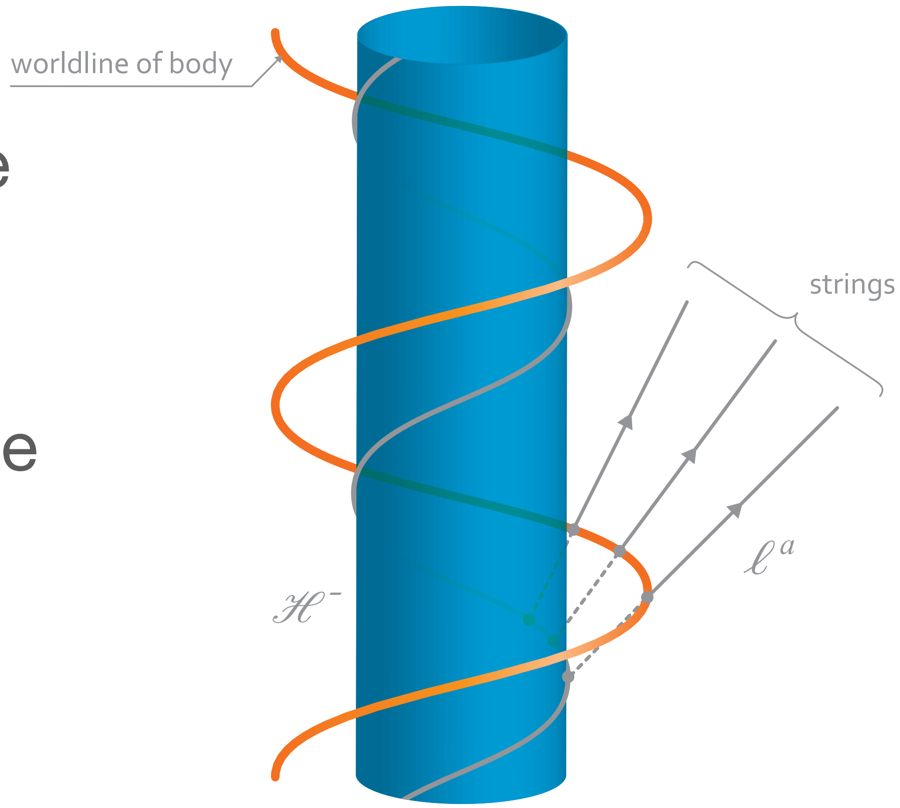
Motivation

$$\boxed{\mathcal{E}^{\text{Kerr}}(h_{ab}) = T_{ab}}$$

- Calculate metric perturbations from point sources in Kerr in a Teukolsky framework.
- Standard CCK reconstruction fails in the presence of sources:
    - No first-order solution in the “shadow” of the source [Ori (2003)].
    - Second-order source nonvanishing everywhere $\Longrightarrow$ No solution anywhere.



Teukolsky operator formalism

- In Kerr, the linearized Einstein equation reduces to a separable wave equation for a curvature scalar

$$\mathcal{E}_{ab}(h) = T_{ab}$$

Linearized Einstein equation about Kerr

Teukolsky operator

Operator identity $\boxed{\mathcal{S}\mathcal{E} = \mathcal{O}\mathcal{T}}$

curvature scalar operator $\mathcal{T}(h) = \psi_0$

$$\begin{aligned} \mathcal{O}(\psi_0) &= \mathcal{O}\mathcal{T}(h) \\ &= \mathcal{S}\mathcal{E}(h) \\ &= \mathcal{S}(T) \end{aligned}$$

Separable wave equation

$$\boxed{\mathcal{O}(\psi_0) = 2\left[(\text{þ} - 4\rho - \bar{\rho})(\text{þ}' - \rho') - (\eth - 4\tau - \bar{\tau}')(\eth' - \tau') - 3\Psi_2\right]\psi_0}$$

Hertz potential

- Typically need metric perturbation h_{ab}, not just Weyl scalar ψ_0.
- **Adjoint identity:** $\mathcal{S}\mathcal{E} = \mathcal{O}\mathcal{T} \longrightarrow \mathcal{E}\mathcal{S}^\dagger = \mathcal{T}^\dagger\mathcal{O}^\dagger$
- Enables metric construction in vacuum

$$\mathcal{O}^\dagger\Phi = 0$$

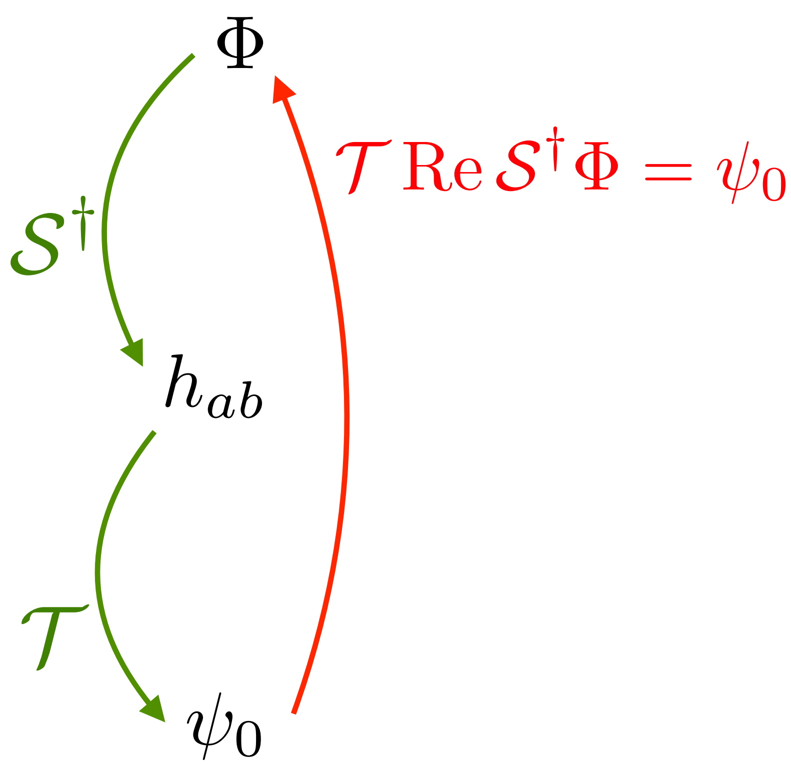
s=-2 Teukolsky equation

Φ = Hertz potential

$$\mathcal{E}_{ab}(h) = 0$$

$$h_{ab} = \mathrm{Re}\,\mathcal{S}^\dagger\Phi$$

Two problems



Hertz potential: $\mathcal{O}^\dagger(\Phi) = \boxed{0}$

metric perturbation: $\mathcal{E}_{ab}(h) = \boxed{0}$

Weyl scalar: $\mathcal{O}(\psi_0) = \boxed{0}$

1. Inversion $\psi_0 \to \Phi$ requires integration of 4th order ordinary differential equation, subject to adjoint Teukolsky for Φ.

2. Vacuum only: h_{ab} in ingoing radiation gauge (IRG), and $\mathcal{E}_{ll}(h^{\mathrm{IRG}}) = 0$

Main result

- We show that h_{ab} satisfying $\mathcal{E}_{ab}(h) = T_{ab}$ can be decomposed as

zero mode

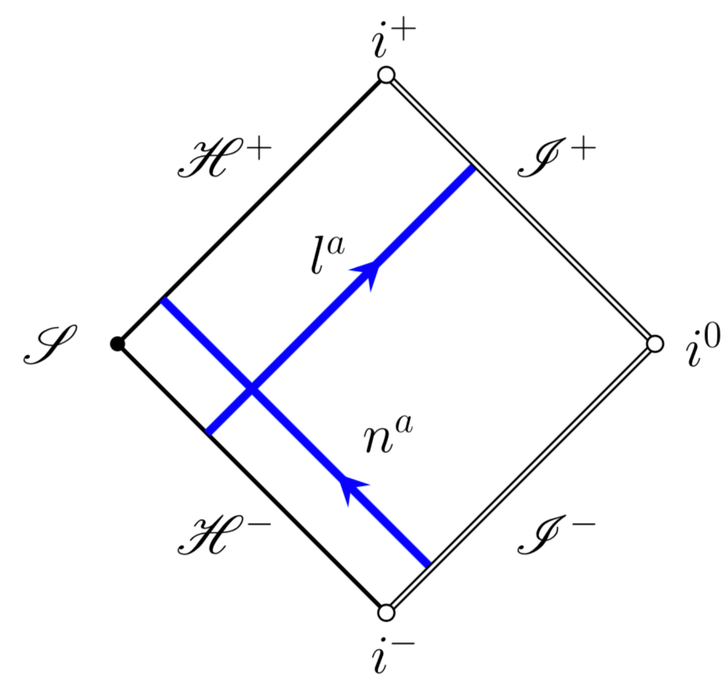
$$h_{ab} = \mathcal{L}_\xi g_{ab} + \dot{g}_{ab} + \boxed{x_{ab}} + \mathrm{Re}\,\mathcal{S}^\dagger_{ab}\Phi$$

Hertz potential satisfying Teukolsky equation with source

gauge

"corrector" tensor:

- obtained by integrating 3 decoupled ordinary differential equations along outgoing principal null directions



Sketch of proof

1. Make a gauge transformation to set $h_{ab}l^b = 0$ (always possible).

2. Pick x_{ab} to cancel off problematic components of T_{ab} ,

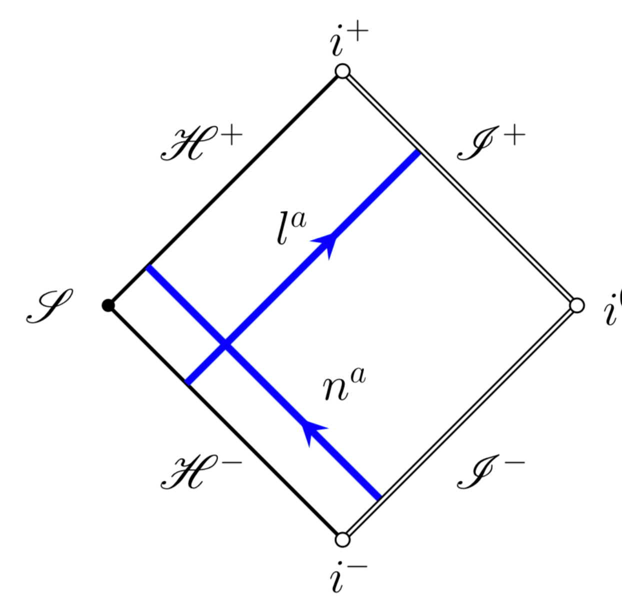
$$(T_{ab} - \mathcal{E}_{ab}(x))\, l^b = 0$$

- Ansatz $x_{ab} = 2m_{(a}\bar{m}_{b)}x_{m\bar{m}} - 2l_{(a}\bar{m}_{b)}x_{nm} - 2l_{(a}m_{b)}x_{n\bar{m}} + l_a l_b x_{nn}$

- Obtain nested set of ODEs along l^a:

(i) $\{\text{þ}(\text{þ} - \rho - \bar{\rho}) + 2\rho\bar{\rho}\}x_{m\bar{m}} = T_{ll}$

$\longrightarrow$ obtain $x_{m\bar{m}}$

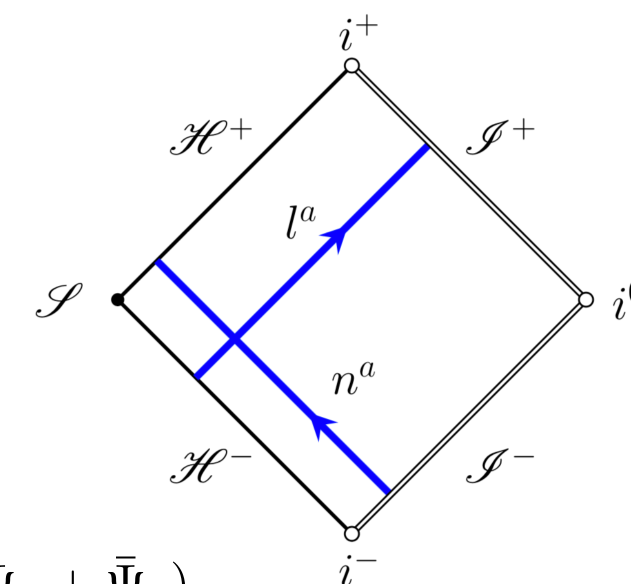


Sketch of proof

(ii)

$$\frac{1}{2}\{\text{þ}(\text{þ}-2\rho)+2\bar{\rho}(\rho-\bar{\rho})\}x_{nm}$$
$$=T_{lm}-\frac{1}{2}\{(\text{þ}+\rho-\bar{\rho})(\eth+\bar{\tau}'-\tau)+2\bar{\tau}'(\text{þ}-2\rho)-(\eth-\tau-\bar{\tau}')\bar{\rho}+2\rho\tau\}x_{m\bar{m}}.$$

$$\longrightarrow \text{obtain } x_{nm}$$



(iii)

$$\frac{1}{2}\{\rho(\text{þ}-\rho)+\bar{\rho}(\text{þ}-\bar{\rho})\}x_{nn}$$
$$=T_{ln}-\frac{1}{2}\{(\eth'+\tau'-\bar{\tau})(\eth-\tau+\bar{\tau}')+(\eth'\eth-\tau\tau'-\bar{\tau}\bar{\tau}'+\tau\bar{\tau})-(\Psi_2+\bar{\Psi}_2)$$
$$+(\text{þ}'-2\rho')\bar{\rho}+(\text{þ}-2\bar{\rho})\rho'+\rho(3\text{þ}'-2\bar{\rho}')+\bar{\rho}'(3\text{þ}-2\rho)$$
$$-2\text{þ}'\text{þ}+2\rho\bar{\rho}'+2\eth'(\tau)-\tau\bar{\tau}\}x_{m\bar{m}}$$
$$-\frac{1}{2}\{(\text{þ}-2\rho)(\eth'-\bar{\tau})+(\tau'+\bar{\tau})(\text{þ}+\bar{\rho})-2(\eth'-\tau')\rho-2\bar{\tau}\text{þ}\}x_{nm}$$
$$-\frac{1}{2}\{(\text{þ}-2\bar{\rho})(\eth-\tau)+(\bar{\tau}'+\tau)(\text{þ}+\rho)-2(\eth-\bar{\tau}')\bar{\rho}-2\tau\text{þ}\}x_{n\bar{m}}$$

$$\longrightarrow \text{obtain } x_{nn}$$

Sketch of proof

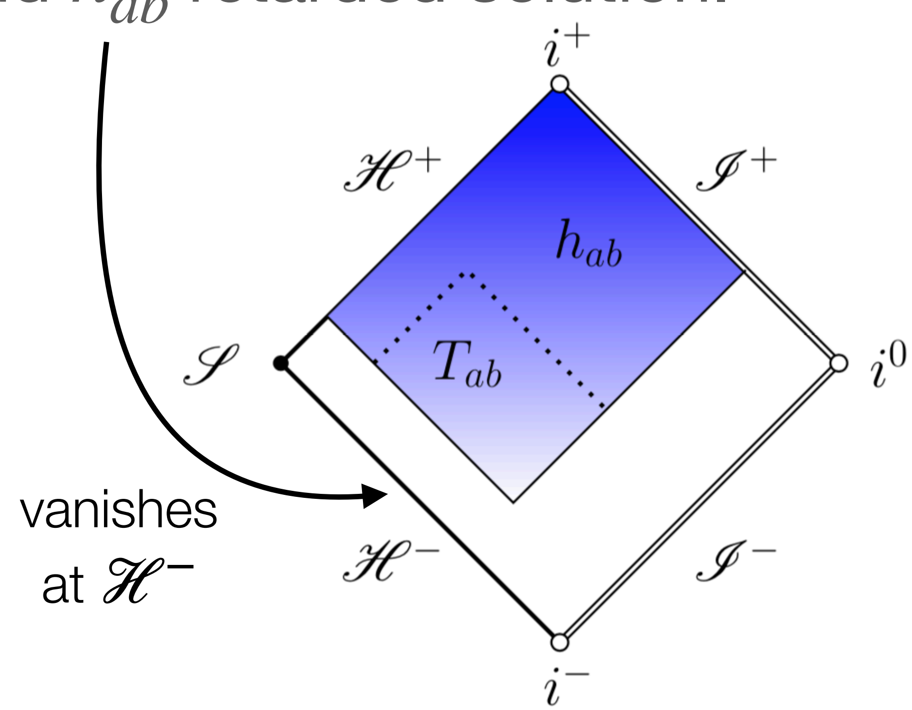
3. Redefine $h_{ab} \to h_{ab} - x_{ab}$

Then: $h_{ab} l^b = 0$ and $\mathcal{E}_{ab}(h) = S_{ab} \equiv T_{ab} - \mathcal{E}_{ab}(x)$

$S_{ab} l^b = 0$

4. Assume T_{ab} smooth with compact support, and h_{ab} retarded solution.

$$
\begin{aligned}
0 &= \mathcal{E}_{ll}(h) \\
&= (\text{þ} - 2\rho)(\text{þ} + \rho - \bar{\rho}) h_{m\bar{m}} \\
&\implies \quad h_{\bar{m}m} = 0 \text{ everywhere}
\end{aligned}
$$



Metric is in IRG automatically!

Sketch of proof

5. Finally, show

$$h_{ab} = \mathrm{Re}\,\mathcal{S}^{\dagger}_{ab}\Phi$$

for some Hertz potential Φ.

I.e.,

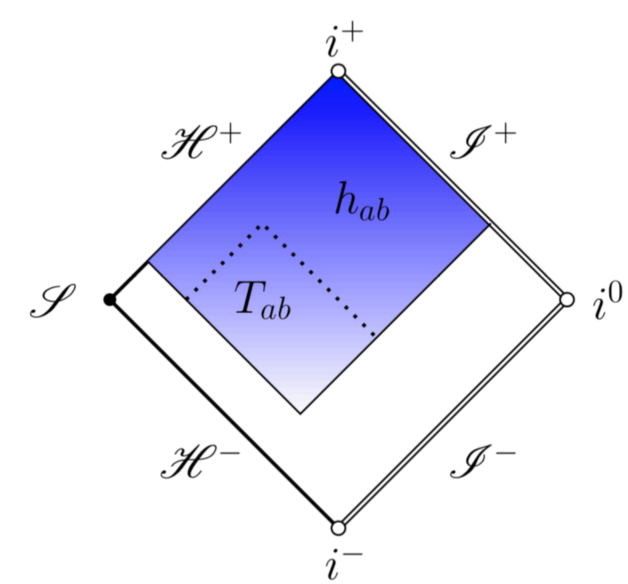
solve this ODE first for Φ

$$h_{\bar{m}\bar{m}} = -\frac{1}{2}(\text{þ} - \rho)(\text{þ} + 3\rho)\Phi,$$

$$h_{\bar{m}n} = -\frac{1}{4}\{(\text{þ} - \rho + \bar{\rho})(\eth + 3\tau) + (\eth - \tau + \bar{\tau}')(\text{þ} + 3\rho)\}\Phi,$$

$$h_{nn} = -\frac{1}{2}(\eth - \tau)(\eth + 3\tau)\Phi + \text{c.c.}$$

these hold near $\mathscr{H}^-$ automatically, and
everywhere by similar arguments to previous slide



∎

Teukolsky equation for $\boldsymbol{\Phi}$

- We established the decomposition,

$$h_{ab} = \mathcal{L}_\xi g_{ab} + \dot{g}_{ab} + x_{ab} + \mathrm{Re}\, \mathcal{S}^\dagger_{ab} \Phi$$

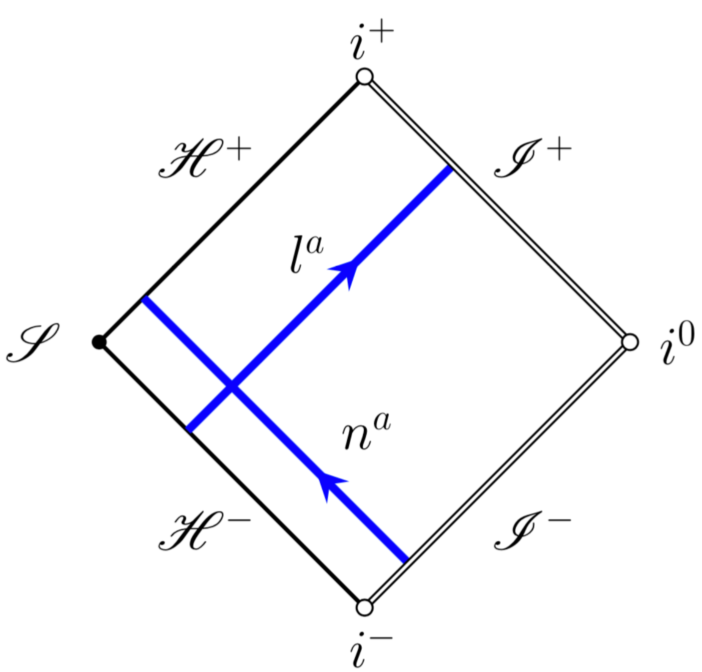
Apply Einstein operator and use operator identity,

$$\implies \quad \mathrm{Re}\, \mathcal{T}^\dagger_{ab} \mathcal{O}^\dagger \Phi = S_{ab}$$

- Thus $\mathcal{O}^\dagger \Phi = \eta$ with source satisfying $\mathrm{Re}\, \mathcal{T}^\dagger_{ab} \eta = S_{ab}$

- Obtain η by integrating $\bar{m}\bar{m}$ component,

$$\frac{1}{4}(\text{þ} - \rho)(\text{þ} - \rho)\eta = S_{\bar{m}\bar{m}}$$



Summary

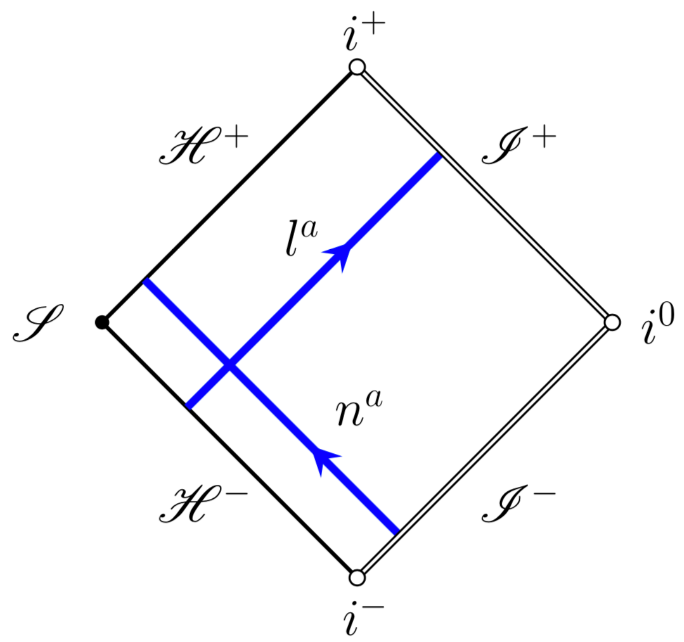
- <u>To obtain metric perturbation solving $\mathcal{E}_{ab}(h) = T_{ab}$</u>

  1. Integrate 3 ODEs along outgoing null geodesics to obtain *corrector tensor* x_{ab}.

  2. Integrate 1 ODE along outgoing null geodesics to obtain source η.

  3. Solve adjoint Teukolsky equation $\mathcal{O}^\dagger \Phi = \eta$

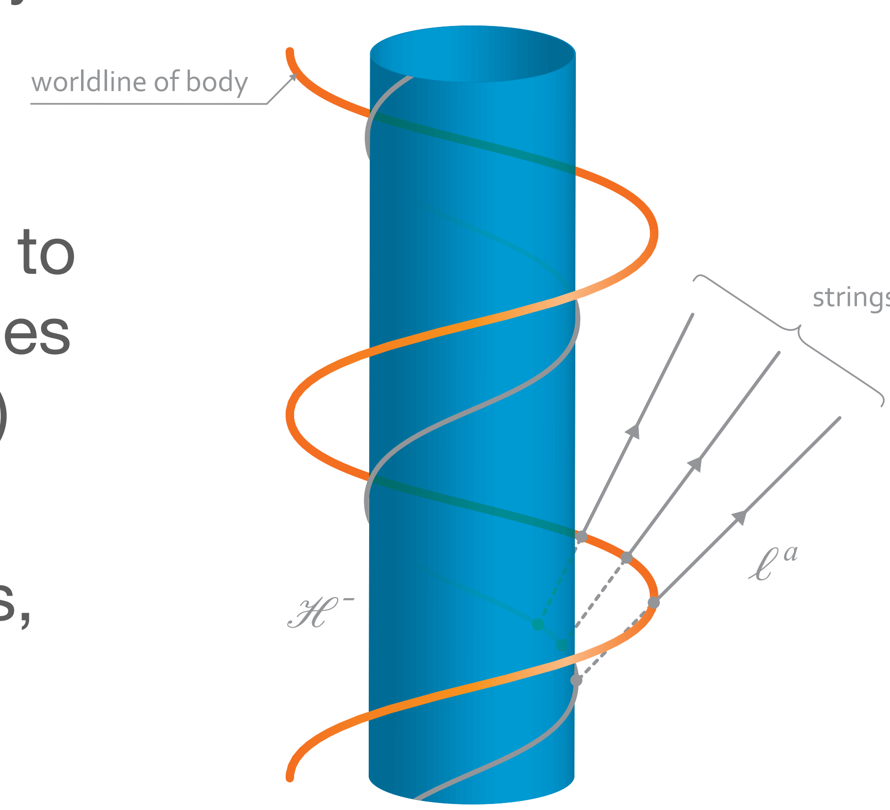
  4. Set $h_{ab} = x_{ab} + \mathrm{Re}\, \mathcal{S}^\dagger_{ab} \Phi$



- Alternatively, start with Teukolsky equation for Weyl scalar ψ_0, obtain Φ by integrating radial ODE, and add corrector tensor at the end.

Comments

- Assumptions on T_{ab}:
    - Proof follows also for distributional T_{ab} (e.g., point particle).
    - Compact support can be relaxed to sufficiently fast decay at $\mathscr{H}^-$.
- Point particle: Algorithm gives distributional *solution* to linearized Einstein, in contrast to standard approaches that fail along distributional string. (Peter's talk, next)
- Other applications to quasinormal mode interactions, perturbative quantum gravity.



Thank you