Orthogonality of Kerr quasinormal modes

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Motivation

- Mode expansions are useful tools as foundations for nonlinear and variational studies.

  E.g., talk by Oleg on modes of global AdS

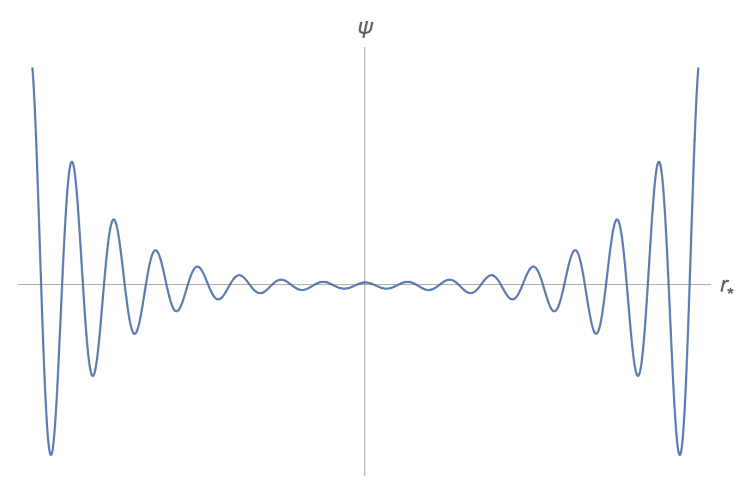
- Normal modes of self-adjoint systems are complete and orthonormal. We can project equations into mode space.

  “bound states”

- With outgoing radiation condition imposed at boundaries, obtain quasinormal modes with $\omega \in \mathbb{C}$.

  “resonance states”

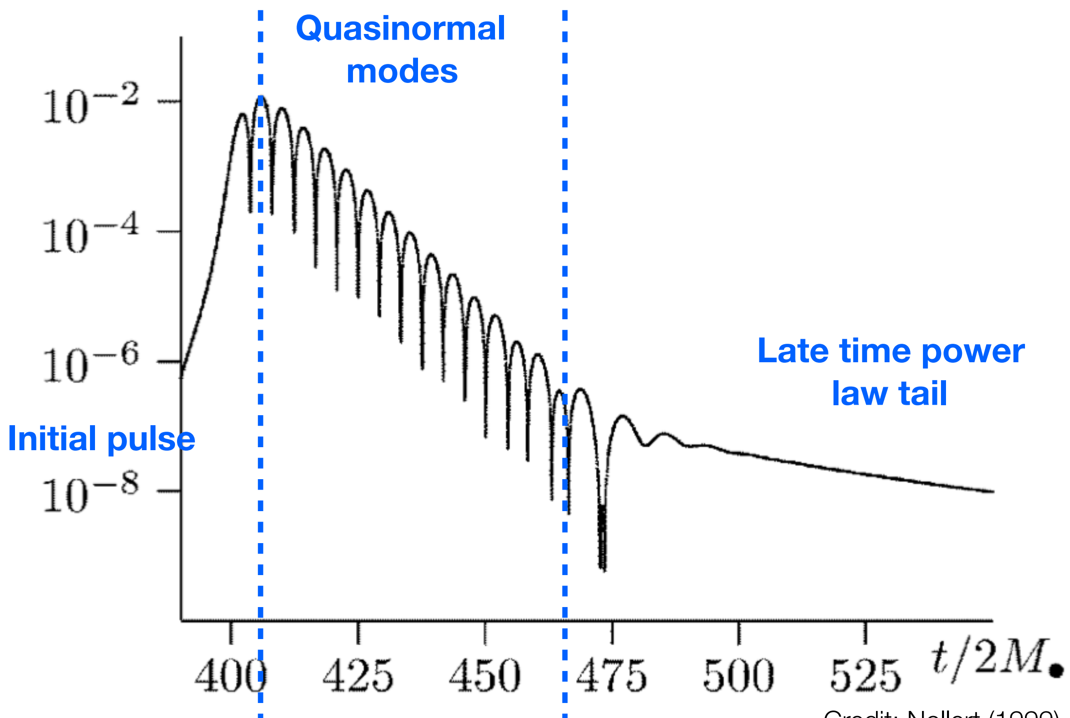
  Physically relevant boundary conditions for black holes and asymptotically flat spacetimes.
  Not in general complete, and not in L^2.



Motivation

- Although not complete, for much of black hole ringdown, quasinormal modes dominate the evolution.

  Would like to develop perturbation theory in terms of quasinormal modes.



Credit: Nollert (1999)

- Possible applications:
  - Near-extreme Kerr
  - Superradiant instability of massive fields in Kerr
  - Kerr-AdS

Summary of results

- Main development: **inner product** $\longrightarrow$ **bilinear form**

  Consider perturbations of a background Kerr spacetime. We define a **symmetric bilinear form** $\langle\langle \cdot , \cdot \rangle\rangle$ on **Weyl scalars** (complex linear in both entries) with the following properties:

  - the time-evolution operator is symmetric with respect to $\langle\langle \cdot , \cdot \rangle\rangle$,

  - $\langle\langle \cdot , \cdot \rangle\rangle$ is finite on quasi-normal modes.

- ***It follows that quasinormal modes with different frequencies are orthogonal with respect to*** $\langle\langle \cdot , \cdot \rangle\rangle$.

- Our bilinear form is based on earlier work by Leung, Liu and Young (1994) on quasinormal modes of open systems.

Outline

1. GHP formalism and Teukolsky equation

2. Lagrangian and Hamiltonian

3. Bilinear form

4. Quasinormal mode orthogonality

5. Extras
    - Relation to Wronskian
    - Excitation coefficients
    - Complex scaling regularization

6. Example: Near-extreme Kerr quasinormal mode orthogonality

Kerr geometry

$$ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 + \frac{4Mar\sin^2\theta}{\Sigma} dt d\phi - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2 - \frac{\Lambda}{\Sigma}\sin^2\theta d\phi^2$$

$$\begin{aligned}
\Delta &= r^2 + a^2 - 2Mr, \\
\Sigma &= r^2 + a^2\cos^2\theta, \\
\Lambda &= (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta
\end{aligned}$$

- Two commuting continuous symmetries. Generated by Killing vectors

$$t^a = (\partial/\partial t)^a, \qquad \varphi^a = (\partial/\partial\phi)^a$$

- Discrete $t-\phi$ reflection symmetry $\quad J : (t, r, \theta, \phi) \to (-t, r, \theta, -\phi)$

  Acting by the push-forward on tensors, J anti-commutes as an operator with symmetries,

$$£_t J = -J£_t, \qquad £_\varphi J = -J£_\varphi.$$

Geroch-Held-Penrose (GHP) formalism

- Kerr is Petrov type D $\Longleftrightarrow$ 2 repeated principle null directions.

  Defines Newman-Penrose null tetrad $(l^a, n^a, m^a, \bar{m}^a)$ aligned with PNDs.

- GHP (1973) developed a framework for writing the Einstein equation such that it transforms covariantly with respect to remaining tetrad freedom.

$$\eta \to \lambda^p \bar{\lambda}^q \eta \quad \Longleftrightarrow \quad \eta \text{ has GHP weights } \{p, q\}$$

- Key GHP covariant operators:

  - **Derivative**: $\Theta_a = \nabla_a - \dfrac{p+q}{2} n^b \nabla_a l_b + \dfrac{p-q}{2} \bar{m}^b \nabla_a m_b$

  - **Lie derivative**: $\text{Ł}_\xi \eta = \pounds_\xi - \dfrac{p+q}{2} n^a \pounds_\xi l_a + \dfrac{p-q}{2} \bar{m}^a \pounds_\xi m_a$

  - t**−**ϕ **reflection**: $\mathcal{J}_*$ = ordinary reflection combined with GHP transformation
    = GHP prime

Teukolsky equation

- Perturbations of Kerr described by ψ_0 or ψ_4. Teukolsky (1972) showed that linearized equations decouple and separate.

$$\left[(þ - 4\rho - \bar{\rho})(þ' - \rho') - (ð - 4\tau - \bar{\tau}')(ð' - \tau') - 3\Psi_2\right] \psi_0 = 0$$

In terms of Θ_a (Bini et al, 2002)

$$\mathcal{O}(\psi_0) \equiv \left[g^{ab}(\Theta_a + 4B_a)(\Theta_b + 4B_b) - 16\Psi_2\right] \psi_0 = 0$$

$$B_a = -(\rho n_a - \tau \bar{m}_a)$$

- Resembles equation for charged scalar field

- $\Psi_2^{-4/3}\psi_4$ satisfies adjoint equation

$$\mathcal{O}^\dagger(\psi_0) = \left[g^{ab}(\Theta_a - 4B_a)(\Theta_b - 4B_b) - 16\Psi_2\right] (\Psi_2^{-4/3}\psi_4) = 0$$

Lagrangian and symplectic form

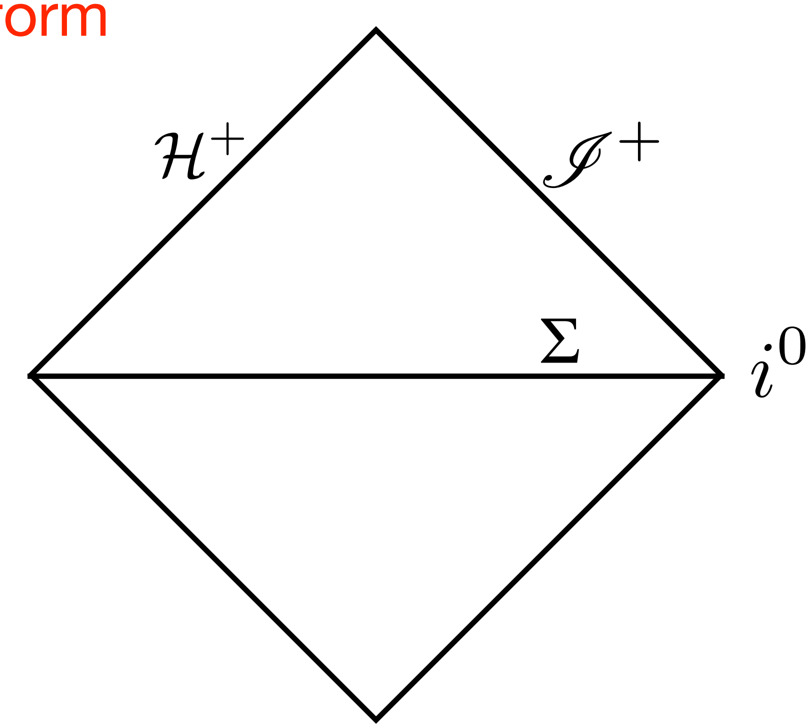
- $\mathscr{O}$ and $\mathscr{O}^\dagger$ equations derive from Lagrangian (Toth, 2018)

$$L(\tilde{\Upsilon}, \Upsilon) = \left[g^{ab}(\Theta_a + 4B_a)\tilde{\Upsilon}(\Theta_b - 4B_b)\Upsilon + 16\Psi_2\tilde{\Upsilon}\Upsilon \right] \epsilon$$

by independently varying $\Upsilon \equiv \Psi_2^{-4/3}\psi_4$, $\tilde{\Upsilon} \equiv \psi_0$.

- Given Cauchy surface Σ and Lagrangian obtain symplectic form

$$\begin{aligned} &W_\Sigma[g; (\Upsilon_1, \tilde{\Upsilon}_1), (\Upsilon_2, \tilde{\Upsilon}_2)] \\ &= \int_\Sigma \epsilon_{dabc} \left[\tilde{\Upsilon}_2(\Theta^d - 4B^d)\Upsilon_1 - \Upsilon_1(\Theta^d + 4B^d)\tilde{\Upsilon}_2 \right. \\ &\qquad \left. - \tilde{\Upsilon}_1(\Theta^d - 4B^d)\Upsilon_2 + \Upsilon_2(\Theta^d + 4B^d)\tilde{\Upsilon}_1 \right] \\ &\equiv \Pi_\Sigma[\tilde{\Upsilon}_2, \Upsilon_1] - \Pi_\Sigma[\tilde{\Upsilon}_1, \Upsilon_2] \end{aligned}$$



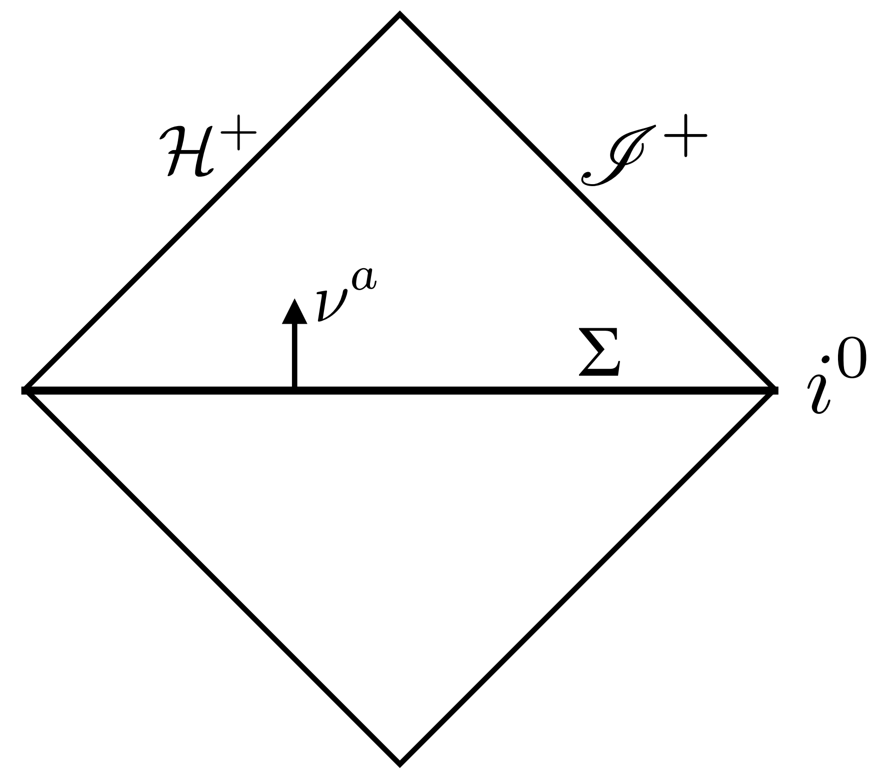
- $\Pi_\Sigma[\tilde{\Upsilon}, \Upsilon]$ conserved on solutions, independent of precise choice of Σ.

Phase space and Hamiltonian

- Boyer-Lindquist slices, with t^a Kerr time-translation Killing vector field. Use GHP covariant Lie derivative $\text{\L}_t$.

- Canonical momentum

$$\varpi = \frac{\partial \mathscr{L}}{\partial(\text{\L}_t \tilde{\Upsilon})} = \sqrt{-h}\nu^a \left(\Theta_a - 4B_a\right) \Upsilon$$



- Legendre transform $\longrightarrow$ Hamiltonian
  $\longrightarrow$ Hamilton's equations

$$\text{\L}_t \begin{pmatrix} \Upsilon \\ \varpi \end{pmatrix} = \mathcal{H} \begin{pmatrix} \Upsilon \\ \varpi \end{pmatrix}$$

with

$$\mathcal{H} = \begin{pmatrix} sM^{1/3}(\Psi_2^{2/3} - 2\xi^a B_a) + N^a(\Theta_a + 2sB_a) & \frac{N}{\sqrt{-h}} \\ -\sqrt{-h}\left[h^{ab}(\Theta_a + 2sB_a)N(\Theta_b + 2sB_b) - 4s^2 N\Psi_2\right] & sM^{1/3}(\Psi_2^{2/3} - 2\xi^a B_a) + (\Theta_a + 2sB_a)N^a \end{pmatrix}$$

Bilinear form

- For GHP scalars $\Upsilon \mathrel{\mathring{=}} \{-4,0\}$, $\tilde{\Upsilon} \mathrel{\mathring{=}} \{4,0\}$ with $\mathscr{O}^\dagger(\Upsilon) = \mathscr{O}(\tilde{\Upsilon}) = 0$, we have the conserved quantity

$$
\begin{aligned}
\Pi_\Sigma[\tilde{\Upsilon}, \Upsilon] &= \int_\Sigma \pi(\tilde{\Upsilon}, \Upsilon) \\
&= \int_\Sigma \epsilon_{dabc} \left[\tilde{\Upsilon}(\Theta^d - 4B^d)\Upsilon - \Upsilon(\Theta^d + 4B^d)\tilde{\Upsilon} \right] \\
&= \int {}^{(3)}e \left(\tilde{\Upsilon}\varpi - \Upsilon\tilde{\varpi} \right)
\end{aligned}
$$

- We would like to define bilinear form on two weight $\{-4,0\}$ scalars.

  Require mapping from $\ker \mathscr{O} \to \ker \mathscr{O}^\dagger$. $t{-}\phi$ reflection

$$
\boxed{\mathcal{O}\Psi_2^{4/3}\mathcal{J}^* = \Psi_2^{4/3}\mathcal{J}^*\mathcal{O}^\dagger}
$$

$t-\phi$ reflection

- Show $\mathcal{O}\Psi_2^{4/3}\mathcal{J}^* = \Psi_2^{4/3}\mathcal{J}^*\mathcal{O}^\dagger$:

$$\begin{aligned}\mathcal{O}\Psi_2^{4/3}\mathcal{J} &= \left[g^{ab}(\Theta_a + 4B_a)(\Theta_b + 4B_b) - 16\Psi_2\right]\Psi_2^{4/3}\mathcal{J} \\ &= \mathcal{J}\left[g^{ab}(\Theta_a + 4B'_a)(\Theta_b + 4B'_b) - 16\Psi_2\right]\Psi_2^{4/3} \\ &= \Psi_2^{4/3}\mathcal{J}\left[g^{ab}(\Theta_a - 4B_a)(\Theta_b - 4B_b) - 16\Psi_2\right] \\ &= \Psi_2^{4/3}\mathcal{J}\mathcal{O}^\dagger\end{aligned}$$

- So $\Psi_2^{4/3}\mathcal{J}^* : \ker\mathcal{O}^\dagger \to \ker\mathcal{O}$

Bilinear form (compact support)

- For $\Upsilon_1, \Upsilon_2 \mathrel{\mathring{=}} \{-4{,}0\}$ of compact support on Σ in $\ker \mathscr{O}^\dagger$, define bilinear form

$$\boxed{\begin{aligned}\langle\langle \Upsilon_1, \Upsilon_2 \rangle\rangle &\equiv \Pi_\Sigma[\Psi_2^{4/3}\mathcal{J}^*\Upsilon_1, \Upsilon_2] \\ &= \int_\Sigma \epsilon_{dabc}\Psi_2^{4/3}\left[(\mathcal{J}^*\Upsilon_1)(\Theta^d - 4B^d)\Upsilon_2 - \Upsilon_2\mathcal{J}^*(\Theta^d - 4B^d)\Upsilon_1\right] \\ &= \int_\Sigma \Psi_2^{4/3}\left[(\mathcal{J}^*\Upsilon_1)\varpi_2 + \Upsilon_2\mathcal{J}^*\varpi_1\right]\end{aligned}}$$

- It can be shown that:
  - (i) $\langle\langle \Upsilon_1, \Upsilon_2 \rangle\rangle = \langle\langle \Upsilon_1, \Upsilon_2 \rangle\rangle$
  - (ii) $\langle\langle \text{Ł}_t\Upsilon_1, \Upsilon_2 \rangle\rangle = \langle\langle \Upsilon_1, \text{Ł}_t\Upsilon_2 \rangle\rangle$
  - (iii) $\langle\langle \Upsilon_1, \Upsilon_2 \rangle\rangle$ is independent of precise choice of Σ

- But $\langle\langle \,\cdot\, , \,\cdot\, \rangle\rangle$ is divergent on quasinormal modes!

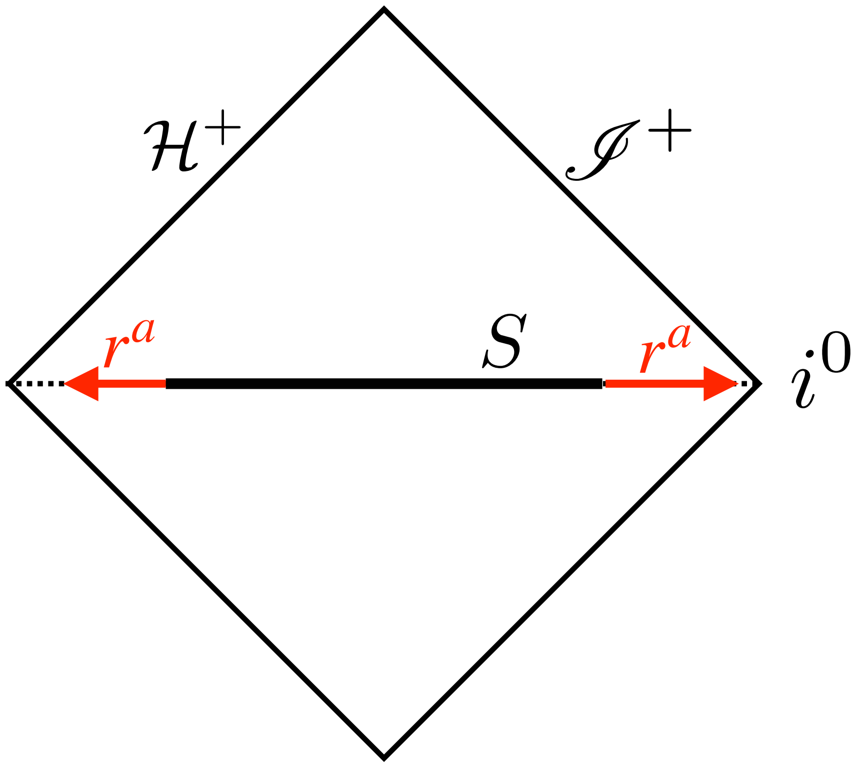
Bilinear form (noncompact support)

- For noncompact support data, try to prove symmetry

$$\langle\langle Ł_t \Upsilon_1, \Upsilon_2 \rangle\rangle = \langle\langle \Upsilon_1, Ł_t \Upsilon_2 \rangle\rangle$$

  on solutions.

  Must keep track of boundary terms.



- On solutions, Cartan's magic formula
  $\implies \pounds_t \pi = d(t \cdot \pi)$ since $d\pi = 0$.

  Integrate over partial Cauchy surface $\implies \int_S \pounds_t \pi(\Psi_2^{4/3} \mathscr{J} \Upsilon_1, \Upsilon_2) = \int_{\partial S} t \cdot \pi(\Psi_2^{4/3} \mathscr{J} \Upsilon_1, \Upsilon_2)$

- Obtain

$$\int_S \pi(\Psi_2^{4/3} \mathcal{J} Ł_t \Upsilon_1, \Upsilon_2) - \int_{\partial S} {}^{(2)}\epsilon N \Psi_2^{4/3} \Upsilon_2 \mathcal{J}[r^a(\Theta_a - 4B_d)\Upsilon_1]$$
$$= \int_S \pi(\Psi_2^{4/3} \mathcal{J} \Upsilon_1, Ł_t \Upsilon_2) - \int_{\partial S} {}^{(2)}\epsilon N \Psi_2^{4/3} (\mathcal{J}\Upsilon_1) r^a(\Theta_a - 4B_d)\Upsilon_2$$

Bilinear form (outgoing radiation)

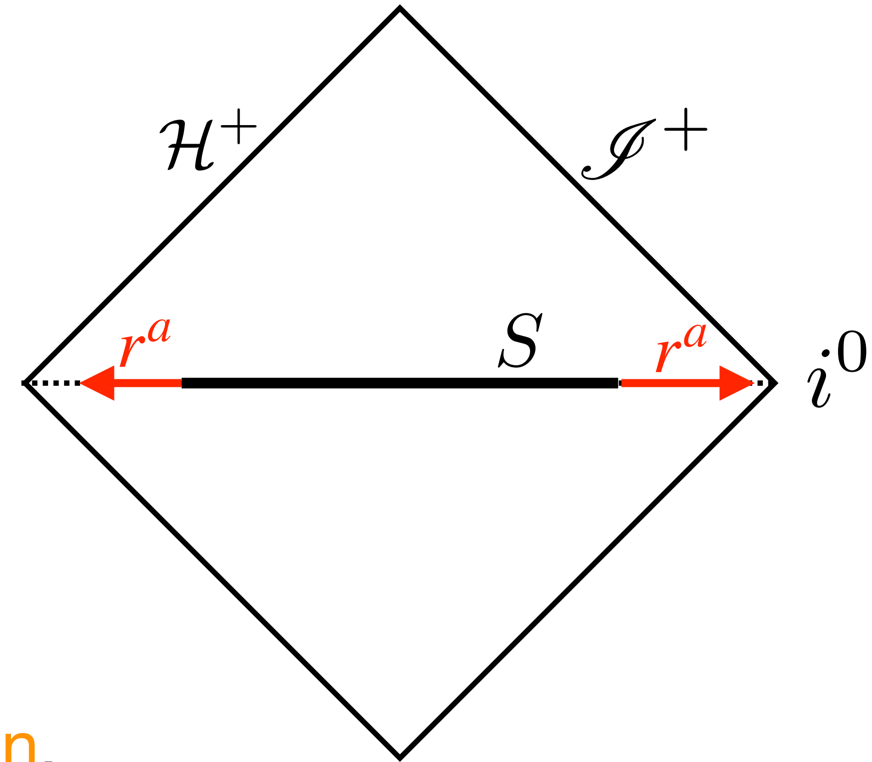
- *Augment bilinear form with boundary terms such that symmetry of* Ł_t *holds.*

- Outgoing radiation condition:

$$\Lambda^{-1/4} r^a (\Theta_a - 4B_a)(\Lambda^{1/4}\Upsilon) \to \frac{1}{\sqrt{-h}}\varpi \quad \text{on } \partial S, \text{ as } S \to \Sigma$$

i.e.,

$$n^a (\Theta_a - 4B_a)(\Lambda^{1/4}\Upsilon) \to 0, \qquad \text{as } r \to r_+$$
$$l^a (\Theta_a - 4B_a)(\Lambda^{1/4}\Upsilon) \to 0, \qquad \text{as } r \to \infty$$



- For Υ_1, Υ_2 satisfying the outgoing radiation condition, define bilinear form

$$\boxed{\langle\langle \Upsilon_1, \Upsilon_2 \rangle\rangle \equiv \lim_{S\to\Sigma} \left\{ \Pi_S[\Psi_2^{4/3} \mathcal{J}\Upsilon_1, \Upsilon_2] + \int_{\partial S} \Psi_2^{4/3} (\mathcal{J}\Upsilon_1)\Upsilon_2 \right\}}$$

Bilinear form (outgoing radiation)

$$\boxed{\langle\langle \Upsilon_1, \Upsilon_2 \rangle\rangle \equiv \lim_{S\to\Sigma} \left\{ \Pi_S[\Psi_2^{4/3}\mathcal{J}\Upsilon_1, \Upsilon_2] + \int_{\partial S} \Psi_2^{4/3}(\mathcal{J}\Upsilon_1)\Upsilon_2 \right\}}$$

- Boundary terms act as a regulator!
- In asymptotic region where outgoing radiation condition holds, the volume integrand becomes exact. Pulled back to surface S,

$$\pi(\Psi_2^{4/3}\mathcal{J}\Upsilon_1, \Upsilon_2) \approx d\left[-{}^{(2)}\epsilon\Psi_2^{4/3}(\mathcal{J}\Upsilon_1)\Upsilon_2\right]$$

volume integrand

boundary integrand

- As we take limit, any additional contribution from larger volume integration exactly counterbalanced by pushing the boundary terms outward.
- Can show that bilinear form satisfies all the other desired properties.

Quasinormal modes

- Quasinormal mode $\boldsymbol{Y} = \begin{pmatrix} \Upsilon \\ \varpi \end{pmatrix}$ with frequency ω, satisfies, on phase space,

$$\mathcal{H}\boldsymbol{Y} = -i\omega\boldsymbol{Y}$$

subject to outgoing radiation condition.

  - Boundary terms in bilinear form precisely cancel divergence in integral to give **finite product between quasinormal modes**.

  - Let $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ be quasinormal modes with frequencies ω_1, ω_2. **Then either $\langle\langle \boldsymbol{Y}_1, \boldsymbol{Y}_2 \rangle\rangle = 0$ or $\omega_1 = \omega_2$.**

    Proof: By symmetry of time-evolution operator,

$$0 = \langle\langle \boldsymbol{Y}_1, \mathcal{H}\boldsymbol{Y}_2 \rangle\rangle - \langle\langle \mathcal{H}\boldsymbol{Y}_1, \boldsymbol{Y}_2 \rangle\rangle = i(\omega_2 - \omega_1)\langle\langle \boldsymbol{Y}_1, \boldsymbol{Y}_2 \rangle\rangle$$

Quasinormal modes

- Separated form of mode solution

$$ {}_s\Upsilon_{\ell m\omega} = e^{-i\omega t + im\phi} {}_sR_{\ell m\omega}(r) {}_sS_{\ell m\omega}(\theta) $$

- Teukolsky showed we get separated angular and radial equations. With Kinnersley tetrad,

$$ \left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + \left(\boxed{K} - \frac{m^2+s^2+2ms\cos\theta}{\sin^2\theta} - a^2\omega^2\sin^2\theta - 2a\omega s\cos\theta\right)\right] {}_sS_{\ell m\omega}(\theta) = 0 $$

$$ \left[\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{d}{dr}\right) + \left(\frac{H^2 - 2is(r-M)H}{\Delta} + 4is\omega r + 2am\omega - \boxed{K} + s(s+1)\right)\right] {}_sR_{\ell m\omega}(r) = 0 $$

with $H \equiv (r^2+a^2)\omega - am$

Angular equation

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)\right.$$
$$\left.+\left(K-\frac{m^2+s^2+2ms\cos\theta}{\sin^2\theta}-a^2\omega^2\sin^2\theta-2a\omega s\cos\theta\right)\right]{}_sS_{\ell m\omega}(\theta)=0$$

- Regular solutions are spin-weighted spheroidal harmonics.

- For fixed s, m, ω, angular functions with different ℓ are orthogonal:

$$\int_0^\pi d\theta\ \sin\theta\,{}_sS_{\ell m\omega}(\theta)\,{}_sS_{\ell' m\omega}(\theta)=\delta_{\ell\ell'}$$

Radial equation

$$\left[\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{d}{dr}\right) + \left(\frac{H^2 - 2is(r-M)H}{\Delta} + 4is\omega r + 2am\omega - K + s(s+1)\right)\right] {}_sR_{\ell m\omega}(r) = 0$$

- Outgoing boundary conditions

$$R^{\rm in} \sim \frac{e^{-ikr_*}}{\Delta^s}, \qquad r_* \to -\infty,$$
$$R^{\rm up} \sim \frac{e^{i\omega r_*}}{r^{2s+1}}, \qquad r_* \to \infty,$$

- Imposing both conditions, obtain discrete set of quasinormal modes with frequency $\omega \in \mathbb{C}$.

- Note: angular and radial equations both depend on ω nonlinearly. Only in phase space, do we have

$$\mathcal{H}\boldsymbol{Y} = -i\omega\boldsymbol{Y}$$

Bilinear form on modes

$$
\begin{aligned}
&\langle\langle \Upsilon_{\ell_1 m_1 \omega_1}, \Upsilon_{\ell_2 m_2 \omega_2} \rangle\rangle \\
&= 8\pi M^{4/3} \delta_{m_1 m_2} e^{-i(\omega_2-\omega_1)t} \lim_{\substack{r_2\to\infty \\ r_1 \to r_+}} \left\{ \int_{r_1}^{r_2} \int_0^\pi dr d\theta \, \frac{\sin\theta}{\Delta^2} S_1 S_2 R_1 R_2 \cdot \right. \\
&\qquad \cdot \left(-\frac{i\Lambda}{\Delta}(\omega_1+\omega_2) + \frac{2iMra}{\Delta}(m_1+m_2) + 2\left[-r - ia\cos\theta + \frac{M}{\Delta}(r^2-a^2) \right] \right) \\
&\qquad \left. + \left[\int_0^\pi d\theta \, \frac{\sqrt{\Lambda}\sin\theta}{\Delta^2} S_1 S_2 R_1 R_2 \right]_{r=r_1} + \left[\int_0^\pi d\theta \, \frac{\sqrt{\Lambda}\sin\theta}{\Delta^2} S_1 S_2 R_1 R_2 \right]_{r=r_2} \right\}.
\end{aligned}
$$

- 2d orthogonality relation: integral does not factorize into 1d integrals, except in special cases ($a \to 0$, near-NHEK, …)
- Cancellations between boundary and volume divergences.

Wronskian

$$\boxed{\mathcal{W}[R_1, R_2] = \Delta^{1+s}(r)\left[R_1(r)\frac{dR_2}{dr} - R_2(r)\frac{dR_1}{dr}\right]}$$

- If R_1, R_2 solutions to radial equation for fixed s, m, ℓ, ω, then Wronskian is independent of r.

- If R_1, R_2 are linearly dependent, then $\mathscr{W}[R_1, R_2] = 0$.

  $\Longrightarrow \mathscr{W}[R_\omega^{\text{in}}, R_\omega^{\text{out}}]$ = 0 at quasinormal frequencies $\omega = \omega_n$.

- What about $d\mathscr{W}[R_\omega^{\text{in}}, R_\omega^{\text{out}}]/d\omega$? At ω_n, gives the norm of the quasinormal mode.

Wronskian

$$\boxed{\mathcal{W}[R_1, R_2] = \Delta^{1+s}(r)\left[R_1(r)\frac{dR_2}{dr} - R_2(r)\frac{dR_1}{dr}\right]}$$

1. Let Υ_1, Υ_2 be GHP scalars in separated form, with the same m, ℓ, ω, but where R_1, R_2 do not necessarily satisfy the radial equation. Then

$$8\pi M^{4/3}\mathcal{W}[R_1, R_2] = \int_{S^2(t,r)} t \cdot \pi(\Psi_2^{4/3}\mathcal{J}\Upsilon_1, \Upsilon_2)$$

2. Let $R^{\text{in}}_\omega, R^{\text{up}}_\omega$ be ingoing, upgoing solutions to the radial equation at frequency ω. Then at a quasinormal frequency ω_n,

$$\frac{d}{d\omega}\mathcal{W}[R^{\text{in}}_\omega, R^{\text{up}}_\omega]\bigg|_{\omega=\omega_n} = \frac{-i}{8\pi M^{4/3}}\langle\langle\Upsilon^{\text{in}}_{\omega_n}, \Upsilon^{\text{up}}_{\omega_n}\rangle\rangle$$

Wronskian

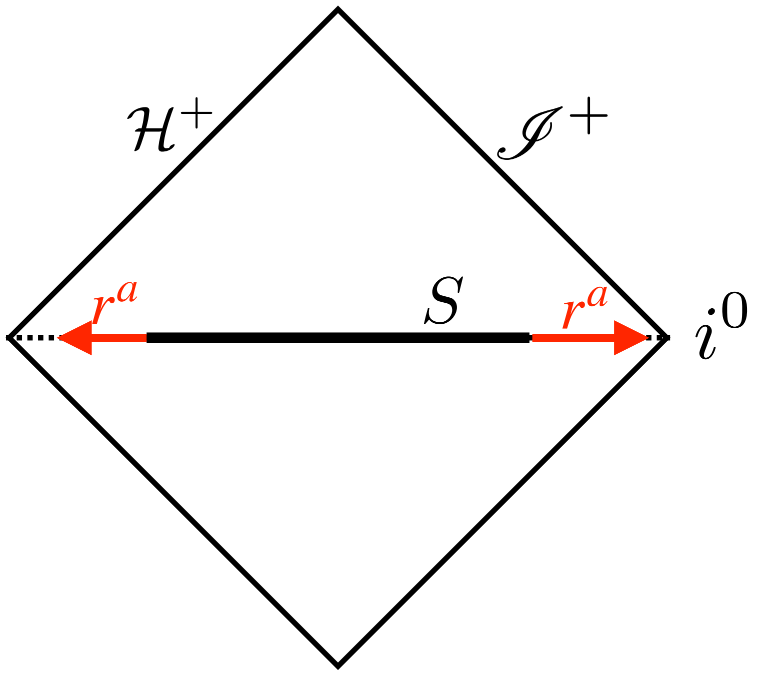
- Sketch of proof of 2: (based on Leung et al, 1994)

Since $d\pi = 0$ on solutions, $d\left(t \cdot \pi\left(\Psi_2^{4/3}\mathcal{J}\Upsilon^{\text{in}}_{\omega_n}, \Upsilon^{\text{up}}_{\omega}\right)\right) = \pounds_t \pi\left(\Psi_2^{4/3}\mathcal{J}\Upsilon^{\text{in}}_{\omega_n}, \Upsilon^{\text{up}}_{\omega}\right)$

$$= -i(\omega - \omega_n)\pi\left(\Psi_2^{4/3}\mathcal{J}\Upsilon^{\text{in}}_{\omega_n}, \Upsilon^{\text{up}}_{\omega}\right)$$

Integrate:

$$\int_{\partial S} t \cdot \pi(\Psi_2^{4/3}\mathcal{J}\Upsilon^{\text{in}}_{\omega_n}, \Upsilon^{\text{up}}_{\omega}) = -i(\omega - \omega_n)\int_S \pi(\Psi_2^{4/3}\mathcal{J}\Upsilon^{\text{in}}_{\omega_n}, \Upsilon^{\text{up}}_{\omega})$$



Differentiate both sides wrt ω, and set $\omega \to \omega_n$:

$$\left.\frac{d}{d\omega}\right|_{\omega=\omega_n} \text{right side} = -i\int_S \pi(\Psi_2^{4/3}\mathcal{J}\Upsilon^{\text{in}}_{\omega_n}, \Upsilon^{\text{up}}_{\omega_n})$$

Wronskian

- <u>Sketch of proof (cont'd)</u>:

$$\frac{d}{d\omega}\bigg|_{\omega=\omega_n} \text{left side} = \int_{\partial S_+} t\cdot\pi\left(\Psi_2^{4/3}\mathcal{J}\Upsilon^{\text{in}}_{\omega_n}, \frac{d}{d\omega}\bigg|_{\omega=\omega_n}\Upsilon^{\text{up}}_{\omega}\right)$$

$$- \frac{d}{d\omega}\bigg|_{\omega=\omega_n} \underbrace{\boxed{\int_{\partial S_-} t\cdot\pi\left(\Psi_2^{4/3}\mathcal{J}\Upsilon^{\text{in}}_{\omega}, \Upsilon^{\text{up}}_{\omega}\right)}}_{\text{Wronskian}} + \int_{\partial S_-} t\cdot\pi\left(\frac{d}{d\omega}\bigg|_{\omega=\omega_n}\Psi_2^{4/3}\mathcal{J}\Upsilon^{\text{in}}_{\omega}, \Upsilon^{\text{up}}_{\omega_n}\right)$$

Combining, $8\pi M^{4/3}\dfrac{d}{d\omega}\mathcal{W}[R^{\text{in}}_\omega, R^{\text{up}}_\omega]\bigg|_{\omega=\omega_n}$

$$= -i\int_S \pi(\Psi_2^{4/3}\mathcal{J}\Upsilon^{\text{in}}_{\omega_n}, \Upsilon^{\text{up}}_{\omega_n})$$

$$-\int_{\partial S_-} t\cdot\pi\left(\frac{d}{d\omega}\bigg|_{\omega=\omega_n}\Psi_2^{4/3}\mathcal{J}\Upsilon^{\text{in}}_{\omega}, \Upsilon^{\text{up}}_{\omega_n}\right) - \int_{\partial S_+} t\cdot\pi\left(\Psi_2^{4/3}\mathcal{J}\Upsilon^{\text{in}}_{\omega_n}, \frac{d}{d\omega}\bigg|_{\omega=\omega_n}\Upsilon^{\text{up}}_{\omega}\right)$$

Asymptotic behaviors of $R^{\text{in}}_\omega, R^{\text{up}}_\omega \implies$ right side reduces to bilinear form in limit $S \to \Sigma$.

$\square$

Excitation coefficients

- Suppose we have compact support initial data $(\Upsilon, \varpi)\,|_{t=0}$

  Then quasinormal mode field response is given by

$$\Upsilon^{\mathrm{QNM}} = \sum_{\ell mn} c_{\ell mn} \Upsilon_{\ell mn}$$

  where

$$\begin{aligned} c_{\ell mn} &= \frac{\langle\langle \Upsilon_{\ell mn}, (\Upsilon, \varpi)\rangle\rangle_{t=0}}{\langle\langle \Upsilon_{\ell mn}, \Upsilon_{\ell mn}\rangle\rangle_{t=0}} \\ &= \frac{1}{\langle\langle \Upsilon_{\ell mn}, \Upsilon_{\ell mn}\rangle\rangle_{t=0}} \int_{\Sigma} {}^{(3)}e \Psi_2^{4/3} [(\mathcal{J}\Upsilon_{\ell mn})\varpi + \Upsilon \mathcal{J} \varpi_{\ell mn}]_{t=0} \\ &= \frac{1}{d\mathcal{W}/d\omega|_{\omega_{\ell mn}}} \frac{1}{2\pi i} \int_0^{2\pi} \int_0^{\pi} \int_{r_+}^{\infty} \frac{\sin\theta}{\Delta^2} e^{-im\phi} S_{\ell mn}(\theta) R_{\ell mn}(r) \left\{ \frac{\Lambda}{\Delta}(\partial_t \Upsilon - i\omega_{\ell mn}\Upsilon) \right. \\ &\qquad \left. + 4\left[\frac{M}{\Delta}(r^2 - a^2) - r - ia\cos\theta\right]\Upsilon + \frac{2Mra}{\Delta}(\partial_\phi \Upsilon + im\Upsilon) \right\}_{t=0} dr d\theta d\phi \end{aligned}$$

- This is precisely result obtained from standard Laplace transform analysis.

Complex scaling

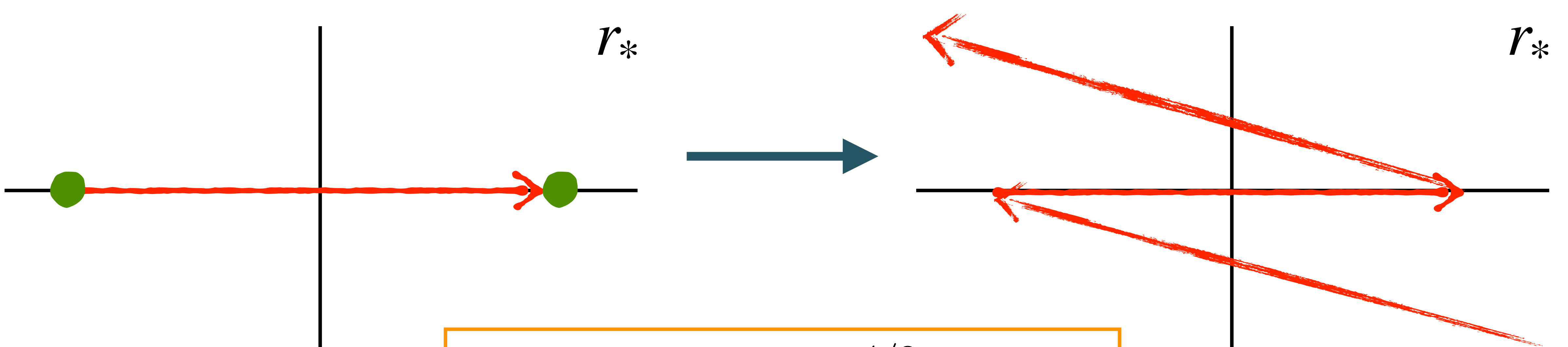
- Numerically, can be tricky to evaluate limit in bilinear form.

$$\langle\langle \Upsilon_1, \Upsilon_2 \rangle\rangle \equiv \lim_{S \to \Sigma} \left\{ \Pi_S[\Psi_2^{4/3} \mathcal{J} \Upsilon_1, \Upsilon_2] + \int_{\partial S} \Psi_2^{4/3} (\mathcal{J} \Upsilon_1) \Upsilon_2 \right\}$$

On modes, volume integrand and boundary terms $\sim e^{\pm i(\omega_1 + \omega_2) r_*}$ as $r_* \to \pm\infty$

$\Longrightarrow$ exponential divergence if $\Im(\omega_1 + \omega_2) < 0$
(Cancellations still give finite result)

- Complexify Σ by deforming into complex-r_* plane such that integrals converge:



$$\boxed{\langle\langle \Upsilon_1, \Upsilon_2 \rangle\rangle = \Pi_{\Sigma_{\mathbb{C}}}[\Psi_2^{4/3} \mathcal{J} \Upsilon_1, \Upsilon_2]}$$

Other bilinear forms: Hertz potentials

- Fundamental identity (Wald, 1978)

$$\mathcal{S}\mathcal{E} = \mathcal{O}\mathcal{T}$$

$\mathscr{T} : \gamma_{ab} \mapsto \psi_0$

Linearized Einstein

Teukolsky

- Adjoint identity

$$\mathcal{E}\mathcal{S}^\dagger = \mathcal{T}^\dagger\mathcal{O}^\dagger$$

- If (ingoing radiation gauge) **Hertz potential** $\psi \mathrel{\mathring{=}} \{-4{,}0\}$ satisfies $\mathscr{O}^\dagger(\psi) = 0$, then
    - $\mathfrak{R}\mathscr{S}^\dagger\Upsilon$ is a real solution to linearized Einstein, and
    - $\Psi_2^{-4/3}\mathscr{T}^\dagger\mathfrak{R}\mathscr{S}^\dagger\psi$ is a solution to $\mathscr{O}^\dagger$ equation, but **not** the same as ψ

Other bilinear forms: Hertz potentials

- If we can find a Hertz potential that generates a given Weyl scalar, then by differentiating, can reconstruct entire metric.

- Suppose Υ_1, Υ_2 generated by (outgoing radiation gauge) Hertz potentials $\tilde{\psi}_1, \tilde{\psi}_2$, i.e.,

$$\Upsilon_i = \Psi_2^{-4/3} \mathcal{T}' \Re \mathcal{S}'^\dagger \Psi_2^{-4/3} \tilde{\psi}_i, \qquad i = 1, 2$$

- Then by repeated application of Prabhu-Wald identity,

$$W_S^{\mathrm{G}}[\gamma, \mathcal{S}^\dagger \Upsilon] = -\Pi_S[\mathcal{T}\gamma, \Upsilon]$$

obtain

$$\begin{aligned}
\langle\langle \Upsilon_1, \Upsilon_2 \rangle\rangle &= -\frac{1}{4} \Pi \left[\tilde{\psi}_2, \Psi_2^{-4/3} \mathcal{T}' \overline{\mathcal{S}'^\dagger \Psi_2^{-4/3} \mathcal{J} \mathcal{T}' \overline{\mathcal{S}'^\dagger \Psi_2^{-4/3} \tilde{\psi}_1}} \right]^* \\
&= -\frac{1}{4} \Pi \left[\tilde{\psi}_2, \Psi_2^{-4/3} \mathcal{J} \text{þ}^4 \left(\bar{\Psi}_2^{-4/3} \text{þ}'^4 \left(\Psi_2^{-4/3} \tilde{\psi}_1 \right) \right) \right]^* \\
&= -\frac{1}{4} \left\langle \left\langle \tilde{\psi}_2, \text{þ}^4 \left(\bar{\Psi}_2^{-4/3} \text{þ}'^4 \left(\Psi_2^{-4/3} \tilde{\psi}_1 \right) \right) \right\rangle \right\rangle_{s=+2}^*
\end{aligned}$$

bilinear form on Hertz potentials

Other bilinear forms: Hertz potential

- Using a Teukolsky-Starobinsky identity, this second argument can be written

$$\text{þ}^4\left(\bar{\Psi}_2^{-4/3}\text{þ}'^4\left(\Psi_2^{-4/3}\tilde{\psi}_1\right)\right) = \eth^4\left(\bar{\Psi}_2^{-4/3}\eth'^4\left(\Psi_2^{-4/3}\tilde{\psi}_1\right)\right) - 9\bar{\text{Ł}}_\xi \text{Ł}_\xi \tilde{\psi}_1$$

algebraic on modes

- So we obtain a relation between bilinear form on Weyl scalars and on Hertz potentials that generate them.

- Similarly, can obtain relation with bilinear form on metric perturbations.

- Aim is to use these relations to go to nonlinear order.

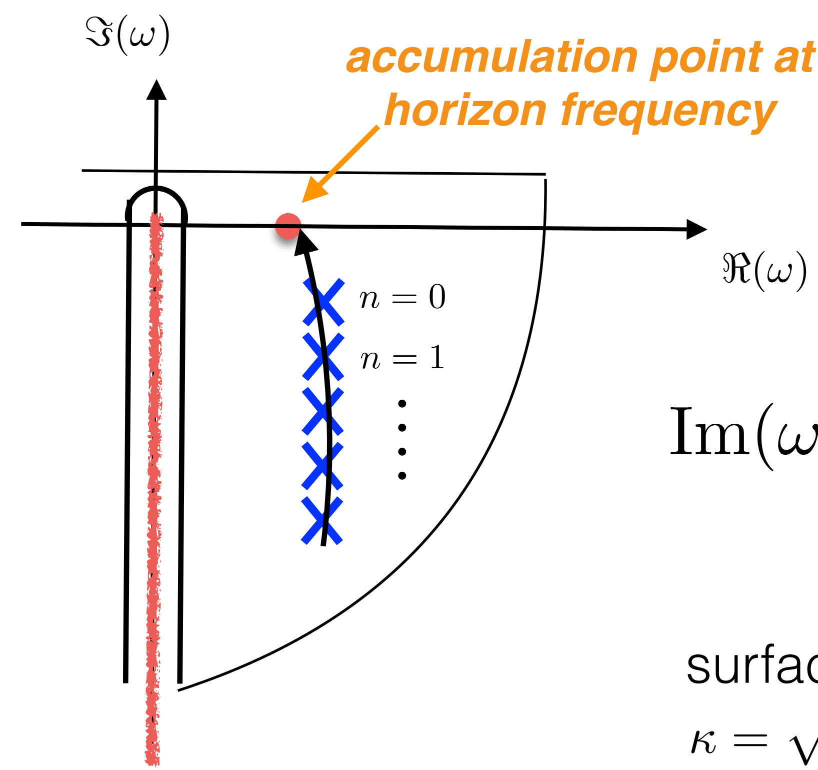
Example: Near-extreme Kerr quasinormal modes

- Near-extreme Kerr has long lived modes. Potential nonlinear turbulent effects, (Yang, Zimmerman, Lehner, 2015).
- Far limit: Extreme Kerr
- Near-NHEK limit:

Extremality parameter $\sigma = \dfrac{r_+ - r_-}{r_+} \to 0$

while holding fixed

$$\bar{t} = t\sigma,$$
$$\bar{x} = \frac{x}{\sigma}, \qquad x = \frac{r - r_+}{r_+}$$
$$\bar{\theta} = \theta,$$
$$\bar{\phi} = \phi - \frac{t}{2M}$$



$\mathrm{Im}(\omega_n) \propto \kappa$

surface gravity
$\kappa = \sqrt{1 - a^2/M^2}$

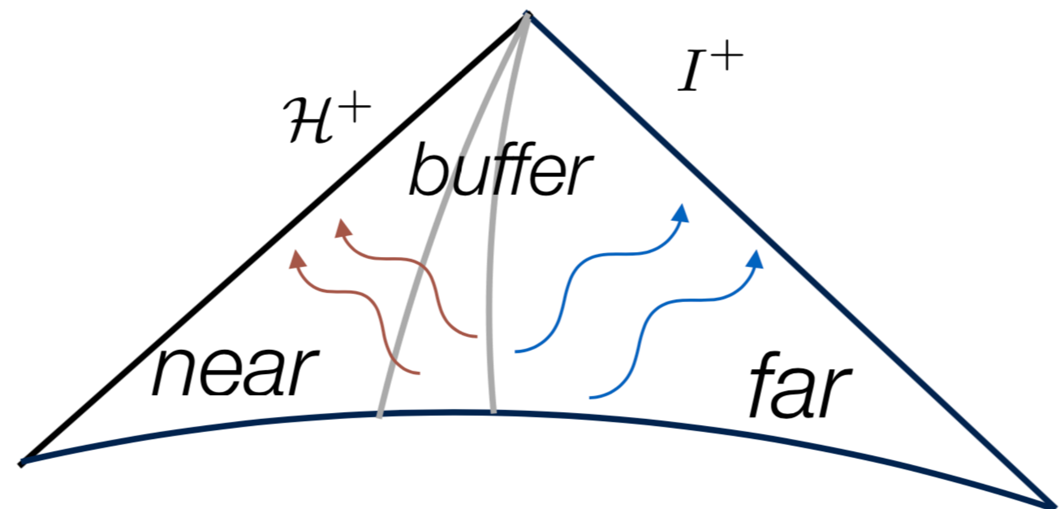
Gives enhanced near-horizon symmetry
$\mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{u}(1)$

Example: Near-extreme Kerr quasinormal modes

- Modes obtained in matched asymptotic expansion

$$\begin{aligned} \text{near-zone: } & x \ll 1 \\ \text{far-zone: } & x \gg \sigma\bar{\omega} \\ \text{overlap region: } & \sigma\bar{\omega} \ll x \ll 1. \end{aligned}$$

$$\text{where } \bar{\omega} = \frac{2M(\omega - m\Omega_H)}{\sigma}$$



- To leading order, spin-weighted spheroidal harmonics evaluated at $\omega = m\Omega_H$.

- Far solution: $\omega = m\Omega_H$ radial solution to extreme Kerr

- Near solution: hypergeometric functions, which reduce to terminating polynomials upon matching

- Matching gives

$$\bar{\omega}_n = -\frac{i}{2}(h_+ + n + im), \qquad h_+ \in \mathbb{R}^+$$

$$\bar{\omega}_n = -\frac{i}{2}(h_- + n + im), \qquad h_- = 1/2 + ir \in \mathbb{C}$$

$$h_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} + K - 2m^2}$$

Example: Near-extreme Kerr quasinormal modes

- Check orthogonality
- Split bilinear form $\langle\langle \Upsilon_1, \Upsilon_2 \rangle\rangle = \langle\langle \Upsilon_1, \Upsilon_2 \rangle\rangle_{\text{near}} + \langle\langle \Upsilon_1, \Upsilon_2 \rangle\rangle_{\text{far}}$
- Near zone:

$$\langle\langle \Upsilon_1, \Upsilon_2 \rangle\rangle_{\text{near}} = 2M^{-2/3} \left[2i \lim_{\epsilon \to 0} \int_{\epsilon}^{c/\sqrt{\sigma}} d\bar{x} \left(\bar{\omega}_1 + \bar{\omega}_2 + 2\hat{e}\hat{A}_{\bar{t}} \right) (x(1+x))^{-3} \, R_1^{\text{near}} R_2^{\text{near}} + \frac{R_1^{\text{near}}(\epsilon) R_2^{\text{near}}(\epsilon)}{\epsilon^2 (1+\epsilon)^2} \right]$$

ϵ-dependent part precisely cancels
amounts to minimal subtraction

- Obtain orthogonality by explicit evaluation.

Conclusions

- We established a bilinear form on Weyl scalars with respect to which Kerr quasinormal modes with different frequencies are orthogonal.

  Construction works in phase space. Relies on type D nature of Kerr and $t-\phi$ reflection symmetry.

- Extensions:

  - Alternative regularization schemes: complex scaling, minimal subtraction

  - Consistency with standard calculations for excitation coefficients

  - Relation of bilinear form on Weyl scalar to bilinear forms on metric perturbations and Hertz potentials

Thank you